

FLOW CURVATURE METHOD APPLIED TO CANARD EXPLOSION

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ABSTRACT. The aim of this work is to establish that the bifurcation parameter value leading to a *canard explosion* in dimension two obtained by the so-called *Geometric Singular Perturbation Method* can be found according to the *Flow Curvature Method*. This result will be then exemplified with the classical Van der Pol oscillator.

1. INTRODUCTION

The classical geometric theory of differential equations developed originally by Andronov [1], Tikhonov [30] and Levinson [23] stated that *singularly perturbed systems* possess *invariant manifolds* on which trajectories evolve slowly, and toward which nearby orbits contract exponentially in time (either forward or backward) in the normal directions. These manifolds have been called asymptotically stable (or unstable) *slow invariant manifolds*¹. Then, Fenichel [11, 12, 13, 14] theory² for the *persistence of normally hyperbolic invariant manifolds* enabled to establish the *local invariance* of *slow invariant manifolds* that possess both expanding and contracting directions and which were labeled *slow invariant manifolds*.

During the last century, various methods have been developed to compute the *slow invariant manifold* or, at least an asymptotic expansion in power of ε .

The seminal works of Wasow [32], Cole [6], O'Malley [25, 26] and Fenichel [11, 12, 13, 14] to name but a few, gave rise to the so-called *Geometric Singular Perturbation Method*. According to this theory, existence as well as local invariance of the *slow invariant manifold* of *singularly perturbed systems* has been stated. Then, the determination of the *slow invariant manifold* equation turned into a regular perturbation problem in which one generally expected the asymptotic validity of such expansion to breakdown [26].

Recently a new approach of n -dimensional singularly perturbed dynamical systems of ordinary differential equations with two time scales, called *Flow Curvature Method* has been developed [17]. In dimension two and three, it consists in considering the *trajectory curves* integral of such systems as *plane* or *space* curves. Based on the use of local metrics properties of *curvature* (*first curvature*) and *torsion* (*second curvature*) resulting from the *Differential Geometry*, this method which does not require the use of asymptotic expansions, states that the location of the points

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¹In other articles the *slow manifold* is the approximation of order $O(\varepsilon)$ of the *slow invariant manifold*.

²The theory of invariant manifolds for an ordinary differential equation is based on the work of Hirsch, *et al.* [19]

where the local *curvature* (resp. *torsion*) of *trajectory curves* of such systems, vanishes, directly provides an approximation of the *slow invariant manifold* associated with two-dimensional (resp. three-dimensional) *singularly perturbed systems* up to suitable order $O(\varepsilon^2)$ (resp. $O(\varepsilon^3)$). This method gives an implicit non intrinsic equation, because it depends on the euclidean metric.

Solutions of “canard” type have been discovered by a group of French mathematicians [2] in the beginning of the eighties while they were studying relaxation oscillations in the classical Van der Pol’s equation (with a constant forcing term) [31]. They observed, within a small range of the control parameter, a fast transition for the amplitude of the limit cycle varying suddenly from small amplitude to a large amplitude. Due to the fact that the shape of the limit cycle in the phase plane looks as a duck they called it “canard cycle”. Hence, they named this new phenomenon “canard explosion”³ and triggered a “duck-hunting”.

Many methods have been developed to analyze “canard” solution such as non-standard analysis [2, 8], matched asymptotic expansions [10], or the blow-up technique [9, 22, 28] which extends the *Geometric Singular Perturbation Method* [11, 12, 13, 14].

Meanwhile, two other geometric approaches have been proposed. The first, elaborated by [4] involves *inflection curves*, while the second makes use of the *curvature* of the trajectory curve, integral of any n -dimensional singularly perturbed dynamical system [16, 17]. This latter, entitled *Flow Curvature Method* will be used in this work in order to compute the bifurcation parameter value leading to a *canard explosion*. Moreover, the total correspondence between the results obtained in this paper for two-dimensional singularly perturbed dynamical systems such as Van der Pol oscillator and those previously established by [2] will enable to highlight another link between the *Flow Curvature Method* and the *Geometric Singular Perturbation Method*.

2. SINGULARLY PERTURBED SYSTEMS

According to Tikhonov [30], Takens [29], Jones [20] and Kaper [21] *singularly perturbed systems* may be defined such as:

$$(1) \quad \begin{aligned} \vec{x}' &= \vec{f}(\vec{x}, \vec{z}, \varepsilon), \\ \vec{z}' &= \varepsilon \vec{g}(\vec{x}, \vec{z}, \varepsilon). \end{aligned}$$

where $\vec{x} \in \mathbb{R}^m$, $\vec{z} \in \mathbb{R}^p$, $\varepsilon \in \mathbb{R}^+$, and the prime denotes differentiation with respect to the independent variable t . The functions \vec{f} and \vec{g} are assumed to be C^∞ functions⁴ of \vec{x} , \vec{z} and ε in $U \times I$, where U is an open subset of $\mathbb{R}^m \times \mathbb{R}^p$ and I is an open interval containing $\varepsilon = 0$.

In the case when $0 < \varepsilon \ll 1$, i.e., is a small positive number, the variable \vec{x} is called *fast* variable, and \vec{z} is called *slow* variable. Using Landau’s notation: $O(\varepsilon^k)$ represents a function f of x and ε such that $f(u, \varepsilon)/\varepsilon^k$ is bounded for positive ε going to zero, uniformly for u in the given domain.

³According to Krupa and Szmolyan [22, p. 312] this terminology has been introduced in chemical and biological literature by Brøns and Bar-Eli [3, p. 8707] to denote a sudden change of amplitude and period of oscillations under a very small range of control parameter.

⁴In certain applications these functions will be supposed to be C^r , $r \geq 1$.

It is used to consider that generally \vec{x} evolves at an $O(1)$ rate; while \vec{z} evolves at an $O(\varepsilon)$ *slow* rate. Reformulating system (1) in terms of the rescaled variable $\tau = \varepsilon t$, we obtain

$$(2) \quad \begin{aligned} \varepsilon \dot{\vec{x}} &= \vec{f}(\vec{x}, \vec{z}, \varepsilon), \\ \dot{\vec{z}} &= \vec{g}(\vec{x}, \vec{z}, \varepsilon). \end{aligned}$$

The dot represents the derivative with respect to the new independent variable τ .

The independent variables t and τ are referred to the *fast* and *slow* times, respectively, and (1) and (2) are called the *fast* and *slow* systems, respectively. These systems are equivalent whenever $\varepsilon \neq 0$, and they are labeled *singular perturbation problems* when $0 < \varepsilon \ll 1$. The label “singular” stems in part from the discontinuous limiting behavior in system (1) as $\varepsilon \rightarrow 0$.

In such case system (2) leads to a differential-algebraic system called *reduced slow system* whose dimension decreases from $m + p = n$ to p . Then, the *slow* variable $\vec{z} \in \mathbb{R}^p$ partially evolves in the submanifold M_0 called the *critical manifold*⁵ and defined by

$$(3) \quad M_0 := \left\{ (\vec{x}, \vec{z}) : \vec{f}(\vec{x}, \vec{z}, 0) = \vec{0} \right\}.$$

When $D_x f$ is invertible, thanks to implicit function theorem, M_0 is given by the graph of a C^∞ function $\vec{x} = \vec{F}_0(\vec{z})$ for $\vec{z} \in D$, where $D \subseteq \mathbb{R}^p$ is a compact, simply connected domain and the boundary of D is an $(p - 1)$ -dimensional C^∞ submanifold⁶.

According to Fenichel theory [11, 12, 13, 14] if $0 < \varepsilon \ll 1$ is sufficiently small, then there exists a function $\vec{F}(\vec{z}, \varepsilon)$ defined on D such that the manifold

$$(4) \quad M_\varepsilon := \left\{ (\vec{x}, \vec{z}) : \vec{x} = \vec{F}(\vec{z}, \varepsilon) \right\},$$

is locally invariant under the flow of system (1). Moreover, there exist perturbed local stable (or attracting) M_a and unstable (or repelling) M_r branches of the *slow invariant manifold* M_ε . Thus, normal hyperbolicity of M_ε is lost via a saddle-node bifurcation of the *reduced slow system* (2).

Definition 1. A “canard” is a solution of a singularly perturbed dynamical system following the attracting branch M_a of the slow invariant manifold, passing near a bifurcation point located on the fold of the critical manifold, and then following the repelling branch M_r of the slow invariant manifold during a considerable amount of time.

Geometrically a *maximal canard* corresponds to the intersection of the attracting and repelling branches $M_a \cap M_r$ of the slow manifold in the vicinity of a non-hyperbolic point. Canards are a special class of solution of singularly perturbed dynamical systems for which normal hyperbolicity is lost.

⁵It corresponds to the approximation of the slow invariant manifold, with an error of $O(\varepsilon)$.

⁶The set D is overflowing invariant with respect to (2) when $\varepsilon = 0$.

3. GEOMETRIC SINGULAR PERTURBATION METHOD

Earliest geometric approaches to *singularly perturbed dynamical systems* have been developed by Cole [6], O'Malley [25, 26], Fenichel [11, 12, 13, 14] for the determination of the *slow manifold* equation.

Geometric Singular Perturbation Method is based on the following assumptions and theorem stated by Nils Fenichel in the middle of the seventies⁷.

3.1. Assumptions.

- (H₁) Functions \vec{f} and \vec{g} are C^∞ functions in $U \times I$, where U is an open subset of $\mathbb{R}^m \times \mathbb{R}^p$ and I is an open interval containing $\varepsilon = 0$.
- (H₂) There exists a set M_0 that is contained in $\{(\vec{x}, \vec{z}) : \vec{f}(\vec{x}, \vec{z}, 0) = 0\}$ such that M_0 is a compact manifold with boundary and M_0 is given by the graph of a C^1 function $\vec{x} = \vec{F}_0(\vec{z})$ for $\vec{z} \in D$, where $D \subseteq \mathbb{R}^p$ is a compact, simply connected domain and the boundary of D is an $(p-1)$ -dimensional C^∞ submanifold. Finally, the set D is overflowing invariant with respect to (2) when $\varepsilon = 0$.
- (H₃) M_0 is normally hyperbolic relative to the *reduced fast system* and in particular it is required for all points $\vec{p} \in M_0$, that there are k (resp. l) eigenvalues of $D_{\vec{x}}\vec{f}(\vec{p}, 0)$ with positive (resp. negative) real parts bounded away from zero, where $k + l = m$.

Theorem 2 (Fenichel's Persistence Theorem). *Let system (1) satisfying the conditions (H₁) – (H₃). If $\varepsilon > 0$ is sufficiently small, then there exists a function $\vec{F}(\vec{z}, \varepsilon)$ defined on D such that the manifold $M_\varepsilon = \{(\vec{x}, \vec{z}) : \vec{x} = \vec{F}(\vec{z}, \varepsilon)\}$ is locally invariant under (1). Moreover, $\vec{F}(\vec{z}, \varepsilon)$ is C^r , and M_ε is C^r $O(\varepsilon)$ close to M_0 . In addition, there exist perturbed local stable and unstable manifolds of M_ε . They are unions of invariant families of stable and unstable fibers of dimensions l and k , respectively, and they are C^r $O(\varepsilon)$ close to their counterparts.*

Proof. See [11], [20] and [21]. □

3.2. Invariance. Generally, Fenichel theory enables to turn the problem for explicitly finding functions $\vec{x} = \vec{F}(\vec{z}, \varepsilon)$ whose graphs are locally *slow invariant manifolds* M_ε of system (1) into regular perturbation problem. Invariance of the manifold M_ε implies that $\vec{F}(\vec{z}, \varepsilon)$ satisfies:

$$(5) \quad \varepsilon D_{\vec{z}}\vec{F}(\vec{z}, \varepsilon) \vec{g}\left(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon\right) = \vec{f}\left(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon\right).$$

Then, plugging the perturbation expansion:

$$\vec{F}(\vec{z}, \varepsilon) = \sum_{i=0}^{N-1} \vec{F}_i(\vec{z}) \varepsilon^i + O(\varepsilon^N)$$

into (5) enables to solve order by order for $\vec{F}(\vec{z}, \varepsilon)$.

⁷For an introduction to Geometric Singular Perturbation Method see [21].

Taylor series expansion for $\vec{f}(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon)$ up to terms of order two in ε leads at order ε^0 to

$$(6) \quad \vec{f}(\vec{F}_0(\vec{z}), \vec{z}, 0) = \vec{0}$$

which defines $\vec{F}_0(\vec{z})$ due to the invertibility of $D_{\vec{x}}\vec{f}$ and the *Implicit Function Theorem*.

At order ε^1 we have:

$$(7) \quad D_{\vec{z}}\vec{F}_0\vec{g}(\vec{F}_0, \vec{z}, 0) = D_{\vec{x}}\vec{f}(\vec{F}_0, \vec{z}, 0)\vec{F}_1 + \frac{\partial \vec{f}}{\partial \varepsilon}(\vec{F}_0, \vec{z}, 0),$$

which yields $\vec{F}_1(\vec{z})$ and so forth.

$$(8) \quad D_{\vec{x}}\vec{f}(\vec{F}_0, \vec{z}, 0)\vec{F}_1 = D_{\vec{z}}\vec{F}_0\vec{g}(\vec{F}_0, \vec{z}, 0) - \frac{\partial \vec{f}}{\partial \varepsilon}(\vec{F}_0, \vec{z}, 0).$$

So, regular perturbation theory enables to build locally *slow invariant manifolds* M_ε . But for high-dimensional *singularly perturbed systems slow invariant manifold* asymptotic equation determination leads to tedious calculations.

Proof. For application of this technique see [14]. \square

3.3. Slow invariant manifold and canards. A manifold of canards is an invariant manifold, where first approximation is M_0 . For two-dimensional singularly perturbed dynamical systems with just one fast variable (x) and one slow variable (y), canards are non generic according to Krupa and Szmolyan [22] and *maximal canards* can only occur in such systems only for discrete values of a control parameter μ . It means that in dimension two a one parameter family of singularly perturbed systems is needed to exhibit canard phenomenon. Because along a canard, the differential $D_x f$ is not always invertible, we can not write the manifold of canards as $x = F(y, \varepsilon)$. Thus, we will suppose that $D_y f$ is invertible and we will try to compute the canard as $y = F(x, \mu, \varepsilon)$. See [5] for a theory of this identification of formal series. We consider the following *singularly perturbed dynamical system*:

$$\begin{aligned} \varepsilon \dot{x} &= f(x, y, \mu, \varepsilon), \\ \dot{y} &= g(x, y, \mu, \varepsilon), \end{aligned}$$

with $x, y \in \mathbb{R}$, i.e. $(m, p) = (1, 1)$ and we suppose that due to the nature of the problem perturbation expansions of the canard and of the canard value read:

$$y = F(x, \varepsilon) = \sum_{i=0}^{N-1} F_i(x) \varepsilon^i + O(\varepsilon^N) \quad \text{and} \quad \mu(\varepsilon) = \sum_{i=0}^{N-1} \mu_i \varepsilon^i + O(\varepsilon^N).$$

According to Eq. (5) invariance of the manifold M_ε reads:

$$(9) \quad \left(\frac{\partial F}{\partial x}(x, \varepsilon) \right) f(x, F(x, \varepsilon), \mu(\varepsilon), \varepsilon) = \varepsilon g(x, F(x, \varepsilon), \mu(\varepsilon), \varepsilon).$$

To avoid technical complications in the computations below, we assume that, at order $O(\varepsilon^0)$, the critical manifold does not depend on the parameter μ .

Indeed,

$$\frac{\partial f}{\partial \mu}(x, F_0(x), \mu_0, 0) = 0$$

Then, solving equation (9) order by order provides at:

Order ε^0

$$(10) \quad \frac{\partial F_0}{\partial x}(x) f(x, F_0(x), \mu_0, 0) = 0 \quad \Leftrightarrow \quad f(x, F_0(x), \mu_0, 0) = 0.$$

because the function $\frac{\partial F_0}{\partial x}(x)$ is almost everywhere non zero. Indeed, the function F_0 is given by the implicit function theorem. In what follows f , g , and their derivatives are evaluated at $(x, F_0(x), \mu_0, 0)$, and F_0 , F_1 and F_2 are evaluated at x .

Order ε^1

$$F_0' \left(\frac{\partial f}{\partial y} F_1 + \frac{\partial f}{\partial \mu} \mu_1 + \frac{\partial f}{\partial \varepsilon} \right) + F_1' f = g.$$

Since according to what has been stated before, we have:

$$(11) \quad F_1 = \frac{\frac{g}{F_0'} - \frac{\partial f}{\partial \varepsilon}}{\frac{\partial f}{\partial y}}$$

A priori, this function is singular at the bifurcation point x_0 of the fast system, because F_0' vanishes at this point. To avoid this singularity in function F_1 , the relation $g(x_0, F(x_0), \mu_0, 0) = 0$ is needed. With an appropriate hypothesis on $\frac{\partial g}{\partial \mu}$, it gives a value for μ_0 .

Higher order The computation can be done with the same arguments. When condition of order k are studied, we have to fix F_k , and to avoid singularity in F_k we have to fix μ_{k-1} . An example will be done in the next paragraph.

3.4. Van der Pol's "canards". Van der Pol system

$$(12) \quad \begin{aligned} \varepsilon \dot{x} &= f(x, y) = x + y - \frac{x^3}{3}, \\ \dot{y} &= g(x, y) = \mu - x, \end{aligned}$$

satisfies Fenichel's assumptions (H_1) – (H_3) except on the points $(x, y) = \pm(1, -\frac{2}{3})$. The critical manifold is the cubic $y = x^3/3 - x$. Thus, the problem is to find a function $y = F(x, \varepsilon)$ whose graph is locally the *slow invariant manifold* M_ε of the Van der Pol system. We write:

$$F(x, \varepsilon) = F_0(x) + \varepsilon F_1(x) + \varepsilon^2 F_2(x) + O(\varepsilon^3)$$

and

$$\mu = \mu_0 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + O(\varepsilon^3).$$

The identification we have to perform is

$$\sum_{i=0} \frac{\partial F_i}{\partial x} \varepsilon^i \left(\sum_{i=0} F_i \varepsilon^i - \left(\frac{x^3}{3} - x \right) \right) = \sum_{i=0} \mu_i \varepsilon^{i+1} - \varepsilon x$$

Then, solving order by order provides at:

Order ε^0

$$F_0(x) = \frac{x^3}{3} - x.$$

Order ε^1

$$(13) \quad F_1(x) = -\frac{x - \mu_0}{x^2 - 1}$$

This function is singular at the fold point $x_0 = 1$ corresponding to the Hopf bifurcation point of the fast system⁸. So, to avoid this singularity in function $F_1(x)$ we pose: $\mu_0 = 1$ and thus we have: $F_1(x) = \frac{-1}{1+x}$.

Order ε^2

$$(14) \quad F_2(x) = \frac{\mu_1 + \frac{(x - \mu_0)(x^2 + 1 - 2\mu_0 x)}{(x^2 - 1)^3}}{x^2 - 1}$$

Taking into account that $\mu_0 = 1$ and, in order to avoid singularity in $F_2(x)$, we find that $\mu_1 = -\frac{1}{8}$ and so $F_2(x) = -\frac{x^2 + 4x + 7}{8(1+x)^4}$.

Order ε^3

Using the same process, a tedious computation (or, better a computation with the help of a computer) leads to $\mu_2 = -\frac{3}{32}$, $\mu_3 = -\frac{173}{1024}$. Thus, the bifurcation parameter value leading to canard solutions reads:

$$\mu = 1 - \frac{1}{8}\varepsilon - \frac{3}{32}\varepsilon^2 + O(\varepsilon^3)$$

Then, for $\varepsilon = 0.01$ one finds again the value obtained numerically by Benoît *et al.* [2, p.99] and Diener [8]:

$$\mu = 0.99874$$

⁸Due to the symmetry of the vector field: $(-x, -y, -\mu) \rightarrow (x, y, \mu)$ the same computation could have been done on the fold point $x_0 = -1$ in the vicinity of which a “canard explosion” also takes place.

The phenomenon of “canard explosion” of Van der Pol system (12) with $\varepsilon = 0.01$ is exemplified on Fig. 1. where the periodic solution has been plotted in red, the critical manifold in black and the positive fixed point in green. Double arrows indicate the *fast* motion while simple arrows indicate the *slow* motion. Exponentially small variations of the parameter value $\mu = 0.99874$ enable to exhibit the transition from relaxation oscillation (a) to small amplitude limit cycles (b) via canard cycles (c). Then, at the parameter value $\mu = 1$ corresponding to the Hopf bifurcation, the canard disappears (d).

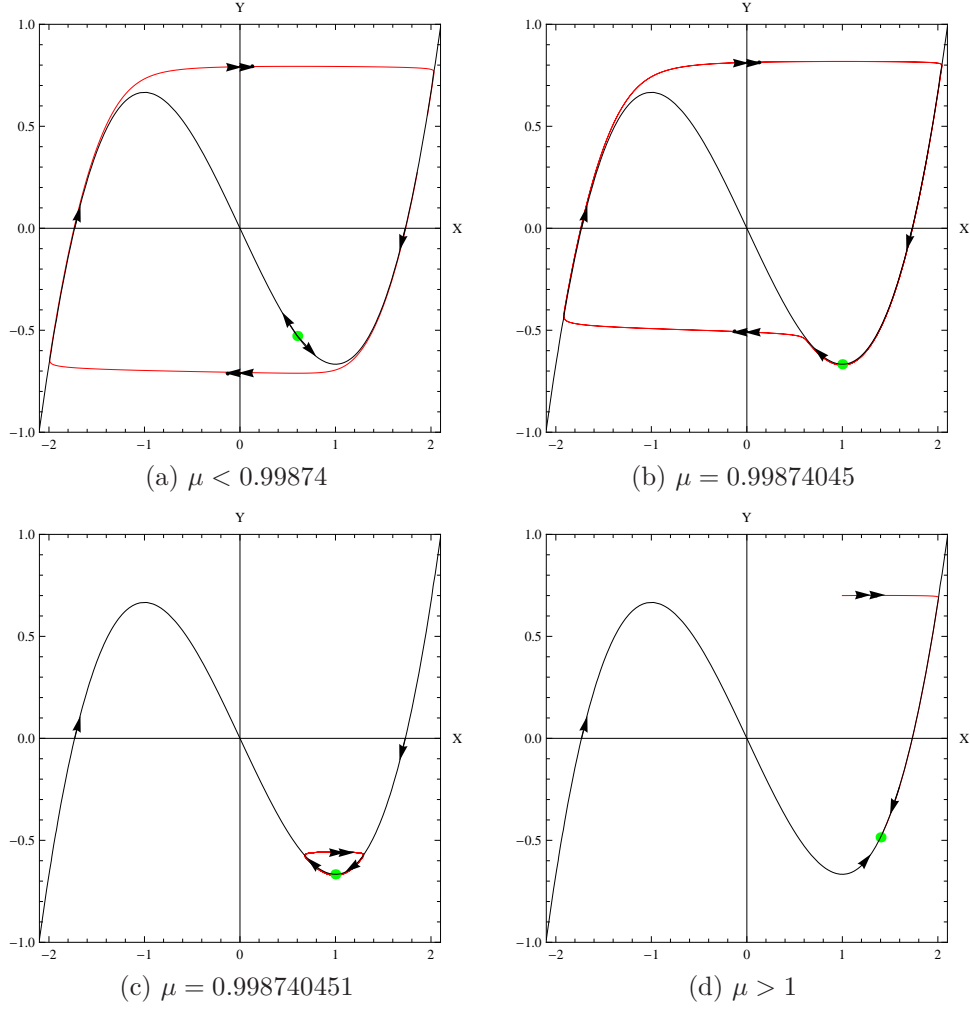


FIGURE 1. Transition from relaxation oscillation to canard explosion.

4. FLOW CURVATURE METHOD

Recently, a new approach called *Flow Curvature Method* and based on the use of *Differential Geometry* properties of *curvatures* has been developed, see [16, 17]. According to this method, the highest *curvature of the flow*, i.e. the $(n-1)^{th}$ *curvature of trajectory curve* integral of n -dimensional dynamical system defines a *manifold* associated with this system and called *flow curvature manifold*. In the case of n -dimensional singularly perturbed dynamical system (1) for which with $\vec{x} \in \mathbb{R}^1$, $\vec{z} \in \mathbb{R}^{n-1}$, i.e. $(m, p) = (1, n-1)$ we have the following result.

Proposition 3. *The location of the points where the $(n-1)^{th}$ curvature of the flow, i.e. the curvature of the trajectory curve \vec{X} , integral of any n -dimensional singularly perturbed dynamical system vanishes, represents its $(n-1)$ -dimensional slow manifold M_ε the equation of which reads*

$$(15) \quad \phi(\vec{X}, \varepsilon) = \dot{\vec{X}} \cdot (\ddot{\vec{X}} \wedge \ddot{\vec{X}} \wedge \dots \wedge \overset{(n)}{\vec{X}}) = \det(\dot{\vec{X}}, \ddot{\vec{X}}, \ddot{\vec{X}}, \dots, \overset{(n)}{\vec{X}}) = 0$$

where $\overset{(n)}{\vec{X}}$ represents the time derivatives up to order n of $\vec{X} = (\vec{x}, \vec{z})^t$.

Proof. For proof of this proposition see [17, p. 185 and next] and below. \square

Remark 4. *First, let's notice that with the Flow Curvature Method the slow manifold is defined by an implicit equation. Secondly, in the most general case of n -dimensional singularly perturbed dynamical system (1) for which $\vec{x} \in \mathbb{R}^m$, $\vec{z} \in \mathbb{R}^p$ the Proposition 3 still holds. In dimension three, the example of a Neuronal Bursting Model (NBM) for which $(m, p) = (2, 1)$ has already been studied by Ginoux et al. [15]. In this particular case, one of the hypotheses of the Tihonov's theorem is not checked since the fast dynamics of the singular approximation has a periodic solution. Nevertheless, it has been established by Ginoux et al. [15] that the slow manifold can all the same be obtained while using the Flow Curvature Method. According to this method, the slow invariant manifold of a three-dimensional singularly perturbed dynamical system for which $(m, p) = (1, 2)$ is given by the 2nd curvature of the flow, i.e. the torsion. In the case of a Neuronal Bursting Model for which $(m, p) = (2, 1)$ it has been stated by Ginoux et al. [15] that the slow manifold is then given by the 1st curvature of the flow, i.e. the curvature. In such a case, the flow curvature manifold is defined by the location of the points where the three-dimensional pseudovector $\dot{\vec{X}} \wedge \ddot{\vec{X}}$ vanishes. This condition leads to a nonlinear system of three equations two of which being linearly independent. These two equations define a curve corresponding to the slow invariant manifold⁹. Thus, one can deduce that for a three-dimensional singularly perturbed dynamical system for which (m, p) the slow manifold is given by the p^{th} curvature of the flow.*

4.1. Invariance. According to Schlomiuk [27] and Llibre et al. [24] the concept of *invariant manifold* has been originally introduced by Gaston Darboux [7, p. 71] in a memoir entitled: *Sur les équations différentielles algébriques du premier ordre et du premier degré* and can be stated as follows.

⁹See also Gilmore et al. [18]

Proposition 5. *The manifold defined by $\phi(\vec{X}, \varepsilon) = 0$ where ϕ is a C^1 in an open set U , is invariant with respect to the flow of (1) if there exists a C^1 function denoted by $\kappa(\vec{X}, \varepsilon)$ and called cofactor which satisfies*

$$(16) \quad L_{\vec{V}}\phi(\vec{X}, \varepsilon) = \kappa(\vec{X}, \varepsilon)\phi(\vec{X}, \varepsilon),$$

for all $\vec{X} \in U$, and with the Lie derivative operator defined as

$$L_{\vec{V}}\phi = \vec{V} \cdot \vec{\nabla}\phi = \sum_{i=1}^n \frac{\partial \phi}{\partial x_i} \dot{x}_i = \frac{d\phi}{dt}.$$

Proof. According to Fenichel's Persistence Theorem (see Th. 2) the slow invariant manifold M_ε may be written as an explicit function $\vec{x} = \vec{F}(\vec{z}, \varepsilon)$, the invariance of which implies that $\vec{F}(\vec{z}, \varepsilon)$ satisfies

$$(17) \quad \varepsilon D_{\vec{z}}\vec{F}(\vec{z}, \varepsilon) \vec{g}(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon) = \vec{f}(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon)$$

We write the slow manifold M_ε as an implicit function by posing

$$(18) \quad \phi(\vec{x}, \vec{z}, \varepsilon) = \vec{x} - \vec{F}(\vec{z}, \varepsilon) = \phi(\vec{X}, \varepsilon).$$

According to Darboux invariance theorem M_ε is invariant if its Lie derivative reads

$$(19) \quad L_{\vec{V}}\phi(\vec{X}, \varepsilon) = \kappa(\vec{X}, \varepsilon)\phi(\vec{X}, \varepsilon).$$

Plugging Eq. (18) into the Lie derivative (19) leads to

$$L_{\vec{V}}\phi(\vec{X}, \varepsilon) = \dot{\vec{x}} - D_{\vec{z}}\vec{F}(\vec{z}, \varepsilon)\dot{\vec{z}} = \kappa(\vec{X}, \varepsilon)\phi(\vec{X}, \varepsilon),$$

which may be written according to Eq. (2) as

$$L_{\vec{V}}\phi(\vec{X}, \varepsilon) = \frac{1}{\varepsilon}(\vec{f}(\vec{X}, \varepsilon) - \varepsilon D_{\vec{z}}\vec{F}(\vec{z}, \varepsilon)\vec{g}(\vec{X}, \varepsilon)) = \kappa(\vec{X}, \varepsilon)\phi(\vec{X}, \varepsilon).$$

Evaluating this Lie derivative in the location of the points where $\phi(\vec{X}, \varepsilon) = 0$, i.e. $\vec{x} = \vec{F}(\vec{z}, \varepsilon)$ leads to

$$L_{\vec{V}}\phi(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon) = \frac{1}{\varepsilon}(\vec{f}(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon) - \varepsilon D_{\vec{z}}\vec{F}(\vec{z}, \varepsilon)\vec{g}(\vec{F}(\vec{z}, \varepsilon), \vec{z}, \varepsilon)) = 0,$$

which is exactly identical to Eq. (18) used by Fenichel. \square

Remark 6. *This last equation for the invariance of the manifold M_ε may be written in a simpler way which implies that $\phi(\vec{x}, \vec{z}, \varepsilon)$ satisfies*

$$(20) \quad \frac{d}{dt}[\phi(\vec{x}, \vec{z}, \varepsilon)] = 0,$$

on the solutions of the differential system.

4.2. Slow invariant manifold. We consider again the following two-dimensional singularly perturbed dynamical system

$$\begin{aligned}\varepsilon \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y),\end{aligned}$$

and we suppose that due to the nature of the problem perturbation expansion reads

$$y = F(x, \varepsilon) = F_0(x) + \varepsilon F_1(x) + \varepsilon^2 F_2(x) + O(\varepsilon^3).$$

According to the *Flow Curvature Method* each function $\vec{F}_i(\vec{z})$ of this perturbation expansion may be found again starting from the *slow manifold implicit equation* (16) as stated in the next result.

Proposition 7. *The functions $F_i(x)$ of the slow invariant manifold associated with a two-dimensional singularly perturbed dynamical system are given by the following expressions*

$$(21) \quad \begin{aligned} F'_n(x) &= \lim_{\varepsilon \rightarrow 0} \left[\frac{1}{n!} \frac{\partial^n a_{10}}{\partial \varepsilon^n} \right] \text{ with } n \geq 0, \\ F_n(x) &= \lim_{\varepsilon \rightarrow 0} \left[\frac{1}{n!} \frac{\partial^{n-1} a_{01}}{\partial \varepsilon^{n-1}} \right] \text{ with } n \geq 1, \end{aligned}$$

where

$$a_{10} = - \left. \frac{\frac{\partial \phi_i}{\partial x}}{\frac{\partial \phi_i}{\partial y}} \right|_{y=F(x, \varepsilon)} \quad \text{and} \quad a_{01} = - \left. \frac{\frac{\partial \phi_i}{\partial \varepsilon}}{\frac{\partial \phi_i}{\partial y}} \right|_{y=F(x, \varepsilon)},$$

and

$$\phi_i(x, y, \varepsilon) = \frac{d^{i-1}}{dt^{i-1}} [\phi_1(x, y, \varepsilon)] \quad \text{with } i = 1, 2, \dots, n,$$

where $\phi_i(x, y, \varepsilon)$ corresponds to the i^{th} order approximation in ε .

Proof. We have that

$$\phi(\vec{X}, \varepsilon) = \left\| \dot{\vec{X}} \wedge \ddot{\vec{X}} \right\| = \det(\dot{\vec{X}}, \ddot{\vec{X}}) = 0 \quad \Leftrightarrow \quad \phi(x, y, \varepsilon) = 0.$$

Since, for a two-dimensional singularly perturbed dynamical systems this *slow manifold equation* is defined by the *second order tensor of curvature*, i.e. by a determinant involving the first and second time derivatives of the vector field \vec{X} , it corresponds to the first order approximation in ε of the *slow manifold* obtained with the *Geometric Singular Perturbation Method*. So, we denote it by

$$\phi_1(\vec{X}, \varepsilon) = \left\| \dot{\vec{X}} \wedge \ddot{\vec{X}} \right\| = \det(\dot{\vec{X}}, \ddot{\vec{X}}) = 0.$$

A *third order tensor of curvature* can be easily given by the time derivative of $\phi_1(\vec{X}, \varepsilon)$. We denote it by

$$\phi_2(\vec{X}, \varepsilon) = \dot{\phi}_1(\vec{X}, \varepsilon) = \det(\dot{\vec{X}}, \ddot{\vec{X}}) = 0.$$

Thus, $\phi_2(\vec{X}, \varepsilon)$ corresponds to the second order approximation in ε . Using the same process, we consider the *slow manifold* $\phi_i(\vec{X}, \varepsilon)$ which corresponds to the i^{th} order approximation in ε .

Writing the *total differential* of the *slow manifold* we obtain

$$(22) \quad d\phi_i(x, y, \varepsilon) = \frac{\partial \phi_i}{\partial x} dx + \frac{\partial \phi_i}{\partial y} dy + \frac{\partial \phi_i}{\partial \varepsilon} d\varepsilon = 0.$$

Replacing in Eq. (23) dy by its *total differential* $dy = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial \varepsilon} d\varepsilon$ yields

$$(23) \quad d\phi_i(x, y, \varepsilon) = \left(\frac{\partial \phi_i}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial F}{\partial x} \right) dx + \left(\frac{\partial \phi_i}{\partial \varepsilon} + \frac{\partial \phi_i}{\partial y} \frac{\partial F}{\partial \varepsilon} \right) d\varepsilon.$$

According to Eq. (21) $\phi_i(x, y, \varepsilon)$ is *invariant* if and only if $d\phi_i(x, y, \varepsilon) = 0$, i.e. if

$$(24) \quad \begin{aligned} \frac{\partial \phi_i}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial F}{\partial x} = 0 & \Leftrightarrow \frac{\partial F}{\partial x} = - \frac{\frac{\partial \phi_i}{\partial x}}{\frac{\partial \phi_i}{\partial y}}, \\ \frac{\partial \phi_i}{\partial \varepsilon} + \frac{\partial \phi_i}{\partial y} \frac{\partial F}{\partial \varepsilon} = 0 & \Leftrightarrow \frac{\partial F}{\partial \varepsilon} = - \frac{\frac{\partial \phi_i}{\partial \varepsilon}}{\frac{\partial \phi_i}{\partial y}}. \end{aligned}$$

By replacing $y = F(x, \varepsilon)$ by its expression in both parts of Eq. (25) and by setting

$$a_{10} = - \frac{\frac{\partial \phi_i}{\partial x}}{\frac{\partial \phi_i}{\partial y}} \Big|_{y=F(x, \varepsilon)} \quad \text{and} \quad a_{01} = - \frac{\frac{\partial \phi_i}{\partial \varepsilon}}{\frac{\partial \phi_i}{\partial y}} \Big|_{y=F(x, \varepsilon)},$$

we have that

$$(25) \quad \begin{aligned} \frac{\partial F(x, \varepsilon)}{\partial x} &= F'_0(x) + \varepsilon F'_1(x) + O(\varepsilon^2) = a_{10}, \\ \frac{\partial F(x, \varepsilon)}{\partial \varepsilon} &= F_1(x) + 2\varepsilon F_2(\varepsilon) + O(\varepsilon^2) = a_{01}. \end{aligned}$$

By using a *recurrence reasoning* it may be easily stated that the functions $F_i(x)$ of the *slow invariant manifold* associated with a two-dimensional singularly perturbed dynamical system are given by the expressions (21). \square

4.3. Van der Pol's "canards". We consider again the Van der Pol system (12). All functions $F_i(x)$ of the perturbation expansion may be deduced from the *slow manifold* equation defined by (16). But, since the determination of $F_3(x)$, i.e. the computation μ_2 requires a *third order tensor of curvature* we consider $\phi_3(\vec{X}, \varepsilon)$ the second time derivative of $\phi_1(\vec{X}, \varepsilon)$ which corresponds to the third order approximation in ε .

Thus, we find at

Order ε^0

$$F'_0(x) = \lim_{\varepsilon \rightarrow 0} [a_{10}] = -1 + x^2,$$

from which one deduces that

$$F_0(x) = \frac{x^3}{3} - x + C_0,$$

where the constant C_0 may be chosen in such a way that the *critical manifold* can be found again ($C_0 = 0$).

Order ε^1

$$(26) \quad F_1(x) = \lim_{\varepsilon \rightarrow 0} [a_{01}] = \frac{\mu_0 - x}{x^2 - 1}.$$

Thus, one find again exactly the same functions $F_1(x)$ as those given by *Geometric Singular Perturbation Method* (14) and of course the same value of the bifurcation parameter $\mu_0 = 1$.

Order ε^2

$$(27) \quad F_2(x) = \frac{\mu_1 + \frac{(x - \mu_0)(x^2 + 1 - 2\mu_0 x)}{(x^2 - 1)^3}}{x^2 - 1}.$$

Taking into account that $\mu_0 = 1$ we find again exactly the same functions $F_2(x)$ as those given by *Geometric Singular Perturbation Method* (15) and of course the same value of the bifurcation parameter $\mu_1 = -1/8$.

Order ε^3

A simple and direct computation leads to $\mu_2 = -\frac{3}{32}$. Thus, the bifurcation parameter value leading to canard solutions reads

$$\mu = 1 - \frac{1}{8}\varepsilon - \frac{3}{32}\varepsilon^2 + O(\varepsilon^3).$$

Then, for $\varepsilon = 0.01$ one finds again the value obtained by Benoît *et al.* [2, p.99] and Diener [8]

$$\mu = 0.99874\dots$$

A program made with Mathematica and available at: <http://ginoux.univ-tln.fr> enables to compute all order of approximations in ε of any two-dimensional singularly perturbed systems.

5. CONCLUSION

Thus, the bifurcation parameter value leading to a *canard explosion* in dimension two obtained by the so-called *Geometric Singular Perturbation Method* has been found again with the *Flow Curvature Method*. This result could be also extended to three-dimensional singularly perturbed dynamical systems such as the 3D-autocatalator in which canard phenomenon occurs.

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